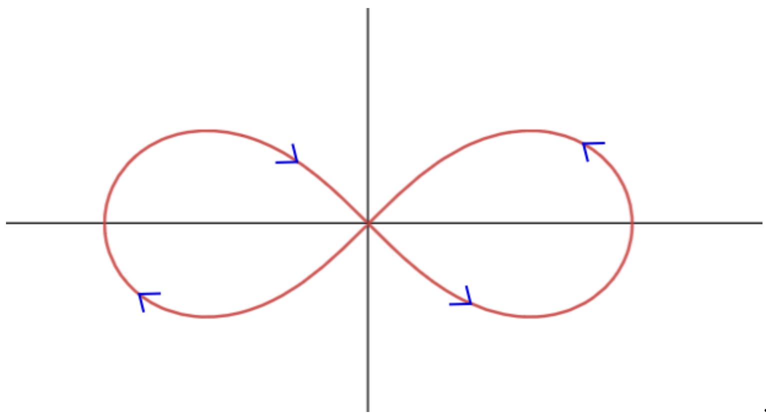
MA 425/525 second midterm review problems

Version as of October 21st.

The second midterm will be in class on Monday October 28th. No notes, books, or electronic devices will be allowed. Most of the exam will be closely based on problems, or on parts of problems, from the list below. Justify your answers. Please let me know if you have a question or find a mistake.

1. What is the order of the zero at 0 of the function $f(z) = \dfrac{z^7}{1 - e^{z^2}}$? What is the radius of convergence of its power series centered at 0?

2. Evaluate $\int_\Gamma e^z \tan 2z \, dz$, where Γ is the contour $|z^2 - 1| = 1$, oriented as in the picture below.



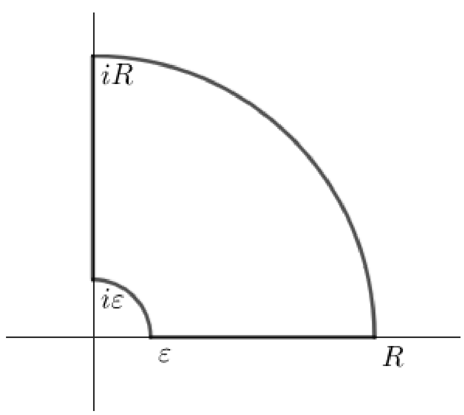
3. Find the residues of $\dfrac{z^3}{\sin z}$, $\dfrac{\cos z}{(z^2+1)^2}$, and $\dfrac{e^{z^n} - 1}{z^m}$ (where n and m are given positive integers) at each of their poles.

4. Find the first three nonzero terms of the Laurent series at zero of the following functions:

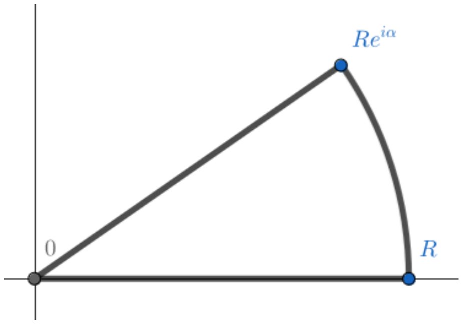
$$\frac{\sin(z^2)}{z}, \qquad \frac{\cos(z^{13}) - 1}{z^7}, \qquad \frac{\operatorname{Log}(1 - \sin(z^3))}{z^2}, \qquad \sin(z)\operatorname{Log}(1-z).$$

(For instance, for ze^{z^2} the answer would be $ze^{z^2} = z + z^3 + \frac{z^5}{2} + \cdots$).

5. Integrate e^{iz}/z over the contour below to find $\int_0^\infty \frac{\sin x}{x} dx$ and $\int_0^\infty \frac{\cos x - e^{-x}}{x} dx$.



6. Let $\gamma > 0$ be a real number and let n be an integer such that $n > \gamma + 1$. Integrate $z^\gamma/(z^n + 1)$ over the contour below, with $\alpha = 2\pi/n$, to find $\int_0^\infty \frac{x^\gamma}{x^n+1} dx$.



Hints: For #1, the order is 5 and the radius is $\sqrt{2\pi}$ by Theorem 1 of Section 2.4. For # 2, use the residue theorem after breaking Γ up into two closed loops. Pay attention to which direction the curve winds around each loop. The residue is most easily computed using Example 4 on page 140. For #3 and #4, you may either learn and use standard Taylor series for exponetial and trig type functions, or derive them by differentiating and using the formula $f(z) = \sum_{n=0}^{\infty} f^{(n)}(0) z^n/n!$. For something like $\sin(z^{13})$ you want to start with the formula for $\sin z$ and subtitute z^{13} for z into it. For # 5, look at Example 8 and Exercise 13 from Section 2.3. For #6, compare your answer with equation (8) from page 165, and it may also help to look at the Example 9 that this equation (8) is based on.